Bernoulli jump-diffusion and discrete SV with jumps models in Bayesian analysis of returns on gas forwards and CO_2 allowance futures

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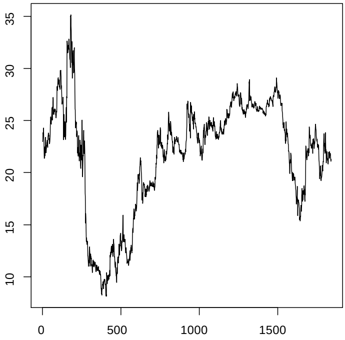
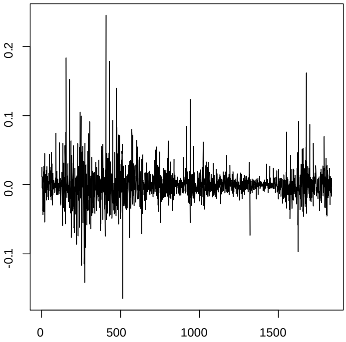
Gas prices - Motivation - Data presentation

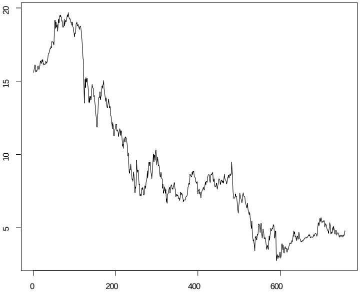
Prices of forward contracts TTF Month 1 (Netherlands TTF Hub), (€/MWh)
Data time range: 21.01.2008-22.04.2015; the number of observations n = 1840

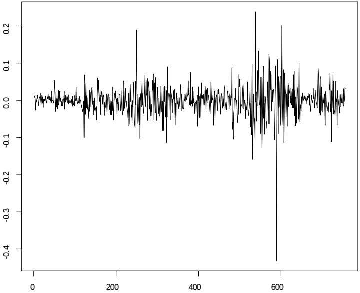




CO2 ALLOWANCES FUTURES CONTRACTS PRICES

The ICE ECX futures. Daily logarithmic returns on CO2 allowances futures expiring on December 16, 2013.
Data time range: January 3, 2011 - December 10, 2013 (n = 756)





Models - DEJD

The Merton model (1976)

S_t - price (Ramezani and Zeng (1998), Kou (2002)):

$$d\left(\ln\left(S_t\right)\right) = \underbrace{\left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t}_{\substack{\text{a pure diffusion}\\ \text{process}}} + \underbrace{d\left(\sum_{i=1}^{N_t}\xi_i\right)}_{\substack{\text{a pure jump}\\ \text{process}}}$$

$$f_{\xi_i}\left(x\right) = p'_D \frac{1}{\eta_D}\exp\left(\frac{1}{\eta_D}x\right)\mathbb{I}_{(-\infty,0)}\left(x\right) + p'_U \frac{1}{\eta_U}\exp\left(-\frac{1}{\eta_U}x\right)\mathbb{I}_{[0,\infty)}\left(x\right)$$

- **The Double Exponential Jump Diffusion model (DEJD)**:

$$y_{t+\Delta} = y_t + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\varepsilon_{t+\Delta}\sqrt{\Delta} + J_{t+\Delta},$$

where $y_t = \ln\left(S_t\right)$, $\varepsilon_{t_i} \sim iid\ N\left(0,1\right)$, $\Delta = 1/252$, $J_{t_i} \sim iid$

$$f_{J_{t_i}}\left(x\right) = p_D \frac{1}{\eta_D}\exp\left(\frac{1}{\eta_D}x\right)\mathbb{I}_{(-\infty,0)}\left(x\right) + p_0\delta_{(0)}\left(x\right) + \\ + p_U \frac{1}{\eta_U}\exp\left(-\frac{1}{\eta_U}x\right)\mathbb{I}_{(0,\infty)}\left(x\right)$$

Models - SVLEPDEJ

$$d\left(\ln\left(S_t\right)\right) = \mu dt + \sqrt{\exp\left(h_t\right)}dW_t^{(1)} + d\left(\sum_{i=1}^{N_t} \xi_i\right)$$

$$dh_t = \kappa_h\left(\theta_h - h_t\right)dt + \sigma_h dW_t^{(2)},\ dW_t^{(1)}dW_t^{(2)} = \rho dt$$

$$f_{\xi_i}\left(x\right) = p'_D \frac{1}{\eta_D}\exp\left(\frac{1}{\eta_D}x\right)\mathbb{I}_{(-\infty,0)}\left(x\right) + p'_U \frac{1}{\eta_U}\exp\left(-\frac{1}{\eta_U}x\right)\mathbb{I}_{[0,\infty)}\left(x\right)$$

- **Stochastic volatility with leverage effect and compound Poisson double exponential jumps model (SVLEPDEJ)**:

$$y_{t+\Delta} = y_t + \mu\Delta + \sqrt{\exp\left(h_t\right)}\sqrt{\Delta}\varepsilon_{t+\Delta}^{(1)} + J_{t+\Delta}$$

$$h_{t+\Delta} = h_t + \kappa_h\left(\theta_h - h_t\right)\Delta + \sigma_h\sqrt{\Delta}\left(\rho\varepsilon_{t+\Delta}^{(1)} + \sqrt{1-\rho^2}\varepsilon_{t+\Delta}^{(2)}\right)$$

$$\varepsilon^{(i)} \sim iid\ N\left(0,1\right),\ \Delta = 1,\ J_{t_i} \sim iid$$

$$f_{J_{t_i}}\left(x\right) = p_D \frac{1}{\eta_D}\exp\left(\frac{1}{\eta_D}x\right)\mathbb{I}_{(-\infty,0)}\left(x\right) + p_0\delta_{(0)}\left(x\right) + p_U \frac{1}{\eta_U}\exp\left(-\frac{1}{\eta_U}x\right)\mathbb{I}_{(0,\infty)}\left(x\right)$$

Models - SVLEPNJ

$$d\left(\ln\left(S_t\right)\right) = \mu dt + \sqrt{\exp\left(h_t\right)} dW_t^{(1)} + d\left(\sum_{i=1}^{N_t} \xi_i\right)$$

$$dh_t = \kappa_h\left(\theta_h - h_t\right) dt + \sigma_h dW_t^{(2)},$$

$$\xi_i \sim iidN\left(\mu_J, \sigma_J^2\right),\ dW_t^{(1)} dW_t^{(2)} = \rho dt$$

- **Stochastic volatility with leverage effect and compound Poisson normal jumps model (SVLEPNJ)**

$$y_{t+\Delta} = y_t + \mu\Delta + \sqrt{\exp\left(h_t\right)}\sqrt{\Delta}\varepsilon_{t+\Delta}^{(1)} + J_{t+\Delta}$$

$$h_{t+\Delta} = h_t + \kappa_h\left(\theta_h - h_t\right)\Delta + \sigma_h\sqrt{\Delta}\left(\rho\varepsilon_{t+\Delta}^{(1)} + \sqrt{1-\rho^2}\varepsilon_{t+\Delta}^{(2)}\right)$$

$\varepsilon_{t_i}^{(1)}, \varepsilon_{t_i}^{(2)} \sim iid\ N\left(0,1\right)$, $\Delta = 1$, $J_{t_i} \sim iid$

$f_{J_{t_i}}\left(x\right) = p_0\delta_{(0)}\left(x\right) + p_1\frac{1}{\sqrt{2\pi\sigma_J^2}}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_J\right)^2}{\sigma_J^2}\right)$

What is a jump? - SVLEPDEJ model

Latent variables $q = (q_{t_1}, ..., q_{t_n})$, $\xi = \left(\xi^D_{t_1}, \xi^U_{t_1} ..., \xi^D_{t_n}, \xi^U_{t_n}\right)$,

$$p(q_{t_i}) = \begin{cases} -1, & p_D \\ 0 & p_0 \\ 1 & p_U \end{cases}, \ \xi^D_{t_i} \sim Exp(\eta_D), \xi^U_{t_i} \sim Exp(\eta_U)$$

- The value of a jump: $J_{t_i} = -\xi^D_{t_i} \cdot \mathbb{I}(q_{t_i} = -1) + \xi^U_{t_i} \cdot \mathbb{I}(q_{t_i} = 1)$
- **Formally, the occurrence of a jump at the *i*-th interval is equivalent to $q_{t_i} \neq 0$.**

The value $q_{t_i} = 0$ means no jump at the *i*-th observation.
The value $q_{t_i} = -1$ means that a jump occurs and its value is negative
The value $q_{t_i} = 1$ means that a jump occurs and its value is positive

- Unfortunately, one does not observe latent variables q.
- Fortunately, the posterior probability of a jump, $P(q_{t_i} \neq 0 | y)$, can be evaluated for each period.

How to detect jumps?
Practice: Let us assume that a jump occurs at the *i*-th period if the posterior probability $P(q_{t_i} \neq 0 | y)$ exceeds an arbitrarily chosen value of 0.5.

Models - Summary

Discrete time scale $t_i = i\Delta$, $i = 0, 1, ..., n$

- The **DEJD** model $\left\{\xi_{t_i}^D\right\} \sim iid\ Exp\left(\eta_D\right)$, $\left\{\xi_{t_i}^U\right\} \sim iid\ Exp\left(\eta_U\right)$

$$y_{t_{i+1}} = y_{t_i} + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\varepsilon_{t_{i+1}}\sqrt{\Delta} + J_{t_{i+1}},$$

$$J_{t_{i+1}}|\left(q_{t_{i+1}} = -1\right) = -\xi_{t_{i+1}}^D,\ J_{t_{i+1}}|(q_{t_{i+1}} = 0) = 0,\ J_{t_{i+1}}|\left(q_{t_{i+1}} = 1\right) = \xi_{t_{i+1}}^U$$

- The **SVLEPDEJ** model: $\left\{\xi_{t_i}^D\right\} \sim iid\ Exp\left(\eta_D\right)$, $\left\{\xi_{t_i}^U\right\} \sim iid\ Exp\left(\eta_U\right)$

$$\begin{aligned} y_{t_{i+1}} &= y_{t_i} + \mu + \sqrt{\exp\left(h_{t_i}\right)}\varepsilon_{t_{i+1}}^{(1)} + J_{t_{i+1}} \\ h_{t_{i+1}} &= h_{t_i} + \kappa_h\left(\theta_h - h_{t_i}\right) + \sigma_h\left(\rho\varepsilon_{t_{i+1}}^{(1)} + \sqrt{1-\rho^2}\varepsilon_{t_{i+1}}^{(2)}\right) \\ J_{t_{i+1}} &= -\xi_{t_{i+1}}^D \cdot \mathbb{I}\left(q_{t_{i+1}} = -1\right) + \xi_{t_{i+1}}^U \cdot \mathbb{I}\left(q_{t_{i+1}} = 1\right) \end{aligned}$$

- The **SVLEPNJ** model $\left\{\xi_{t_i}\right\} \sim iid\ N\left(\mu_J, \sigma_J^2\right)$

$$\begin{aligned} y_{t_{i+1}} &= y_{t_i} + \mu + \sqrt{\exp\left(h_{t_i}\right)}\varepsilon_{t_{i+1}}^{(1)} + J_{t_{i+1}} \\ h_{t_{i+1}} &= h_{t_i} + \kappa_h\left(\theta_h - h_{t_i}\right) + \sigma_h\left(\rho\varepsilon_{t_{i+1}}^{(1)} + \sqrt{1-\rho^2}\varepsilon_{t_{i+1}}^{(2)}\right) \\ J_{t_{i+1}} &= 0 \cdot \mathbb{I}\left(q_{t_{i+1}} = 0\right) + \xi_{t_i} \cdot \mathbb{I}\left(q_{t_{i+1}} = 1\right) \end{aligned}$$

Bayesian Approach

$y = (y_1, ..., y_n)$ the observed data
θ the vector of unknown parameters

$p(\theta)$ the prior density

$p(y|\theta)$ the sampling density

$p(y, \theta) = p(y|\theta)\, p(\theta)$ the Bayesian model

$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$ the posterior density

SVLEPDEJ - Bayesian Approach

The vector of unknown parameters:

$$\underbrace{(\mu,\kappa_h,\theta_h,\sigma_h,\rho,\eta_D,\eta_U,p_D,p_0,p_U}_{=\theta},\underbrace{h_{t_0},\ldots,h_{t_{n-1}}}_{=h},\underbrace{q_{t_1},\ldots,q_{t_n}}_{=q},\underbrace{\xi^D_{t_1},\xi^U_{t_1}\ldots,\xi^D_{t_n},\xi^U_{t_n})}_{=\xi}$$

Parametrisation: $(\sigma_h,\rho)\rightarrow(\phi_h,\omega_h)$; $\phi_h=\sigma_h\rho$, $\omega_h=\sigma_h^2\left(1-\rho^2\right)$

The prior structure:

$\mu\sim N(0,10)$, $\kappa_h\sim N(1,6)\,\mathbb{I}_{(0,2)}$, $\theta_h\sim N(0,10)$, $\omega_h\sim IG\left(3,\frac{1}{20}\right)$,

$\phi_h\,|\omega_h\,\sim N\left(0,\frac{1}{2}\omega_h\right)$,

$p(\eta_D)\sim IG(1.86,0.43)$, $p(\eta_U)\sim IG(1.86,0.43)$,

$(p_D,p_0,p_U)\sim Dirichlet(1,1,1)$

Latent variables: $p\left(q_{t_i}\,|\theta\right)=\begin{cases}-1, & p_D\\ 0 & p_0\\ 1 & p_U\end{cases}$, $\xi^D_{t_i}\,|\theta\sim Exp(\eta_D)$,

$\xi^U_{t_i}\,|\theta\sim Exp(\eta_U)$

Values of jumps: $J_{t_i}=-\xi^D_{t_i}\cdot\mathbb{I}\left(q_{t_i}=-1\right)+\xi^U_{t_i}\cdot\mathbb{I}\left(q_{t_i}=1\right)$

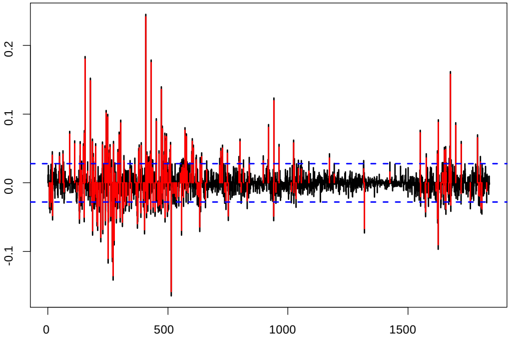
The Bayesian model: $p(y,\theta,h,q,\xi)=p\left(y\,|\theta,h,q,\xi\right)p(\theta,h,q,\xi)$

Empirical Results - DEJD model - gas (TTF Month 1)

Posterior means and standard deviations of parameters

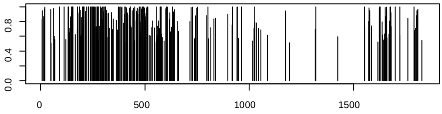
θ	$E(\cdot \mid y)$	$D(\cdot \mid y)$
λ	96.702056	16.6284778
μ'	-0.1573165	0.1121830
σ	0.1962161	0.007352902
p_U'	0.475499	0.0525424
η_D	0.02625504	0.0029278
η_U	0.03415832	0.0035563

Empirical Results - DEJD model - Gas (TTF Month 1)

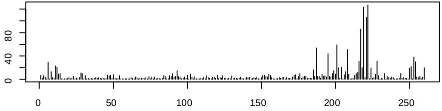


The logarithmic returns. The red colour represents the posterior means of jump values. The blue dashedlines denote values of *sample mean* $+c \cdot$ *sample standard deviation*, where $c = -1.143$ for negative returns and $c = 1.145$ for positive returns.

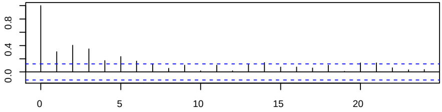
The jump clustering phenomenon



1. The posterior probabilities of jumps.



2. The waiting times series for consecutive jumps
3. The autocorrelation function calculated for the series of times between consecutive jumps.

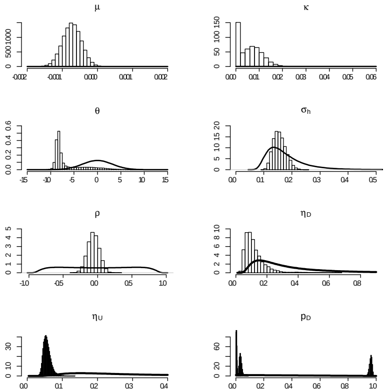


- The results may indicate the jump clustering phenomenon.

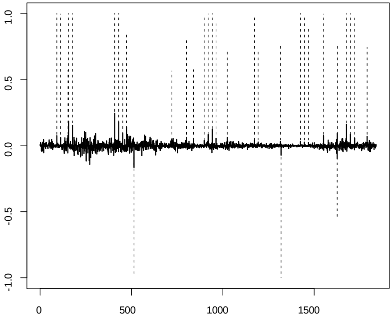
Empirical Results - SVLEPDEJ - Gas (TTF Month 1)

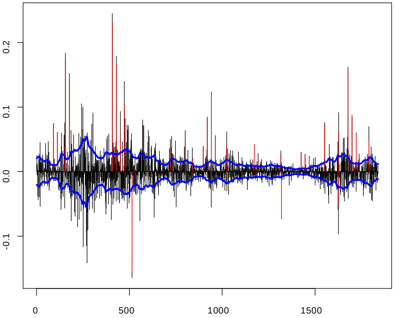
$$y_{t_{i+1}} = y_{t_i} + \mu + \sqrt{\exp\left(h_{t_i}\right)}\varepsilon^{(1)}_{t_{i+1}} - \xi^{D}_{t_{i+1}} \cdot \mathbb{I}\left(q_{t_{i+1}} = -1\right) + \xi^{U}_{t_{i+1}} \cdot \mathbb{I}\left(q_{t_{i+1}} = 1\right)$$
$$h_{t_{i+1}} = h_{t_i} + \kappa_h\left(\theta_h - h_{t_i}\right) + \sigma_h\left(\rho\varepsilon^{(1)}_{t_{i+1}} + \sqrt{1-\rho^2}\varepsilon^{(2)}_{t_{i+1}}\right)$$

θ	$E\left(\cdot\middle|x\right)$	$D\left(\cdot\middle|x\right)$
μ	-0.00073	0.00027
κ_h	0.00599	0.00480
θ_h	-6.43938	3.06274
σ_h	0.15558	0.02214
ρ	-0.05999	0.08403
η_D	0.12019	0.07006
η_U	0.05643	0.01092
p_D	0.00458	0.00308
p_0	0.96085	0.00959
p_U	0.03562	0.00876



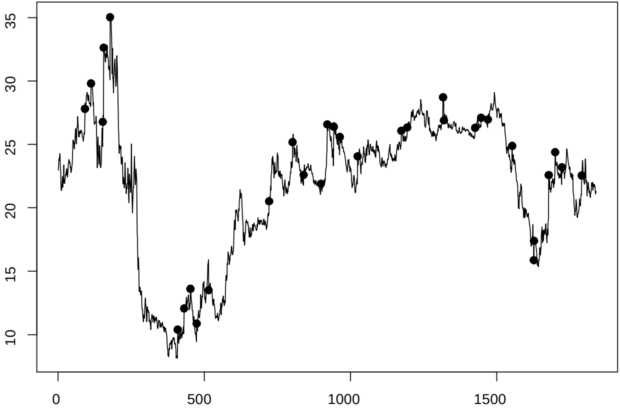
Empirical Results - SVLEPDEJ - Gas (TTF Month 1)





Logarithmic returns (solid line) and posterior probabilities of jumps (dotted line).
Logarithmic returns (black), posterior means of jump values (red) and SV (blue).

Analysis of jumps - SVLEPDEJ - Gas (TTF Month 1)



Analysis of jumps - Gas (TTF Month 1)

	Mon.	Tue.	Wed.	Thu.	Fri.	$\Sigma_{D,U}$		$\Sigma_{D,U}$
$DEJD_{down}$	36	25	23	16	28	132	$VPT1_{down}$	32
$DEJD_{up}$	41	28	23	21	19	128	$VPT1_{up}$	32
$DEJD_{\Sigma=}$						**260**	$VPT1_{\Sigma=}$	**64**
$SVLEPDEJ_{down}$	1	1	0	0	0	2	$VPT1_{down}$	130
$SVLEPDEJ_{up}$	15	7	2	2	2	28	$VPT1_{up}$	130
$SVLEPDEJ_{\Sigma=}$						**30**	$VPT1_{\Sigma=}$	**260**
$SVLEPNJ_{down}$	2	1	0	1	1	5	RFR_{down}	41
$SVLEPNJ_{up}$	12	7	2	2	2	25	RFR_{up}	61
$SVLEPNJ_{\Sigma=}$						**30**	$RFR_{\Sigma=}$	**102**

Janczura et al. (2013):

- VPT1 (2.5% highest and 2.5% lowest returns are treated as jumps)
- VPT2 (10% highest and 10% lowest returns are treated as jumps),
- RFR (returns exceeding the mean return by three standard deviations are treated as jumps, with the jump values removed one by one ('recursive filter')).

Analysis of jumps - Gas (TTF Month 1)

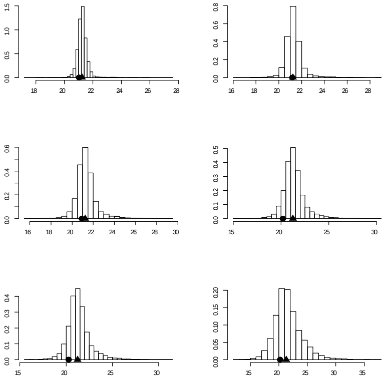
	First trading day of a month	**Last trading day of a month**	**After a break**
DEJD	39 jumps 15% jumps	19 jumps 6% jumps	89 jumps 34% jumps
SVLEPDEJ	23 jumps 79.31% jumps 26.44% months	0 jumps 0% jumps 0% months	18 jumps 58.62% jumps $<$ 4.9% weekends
SVLEPNJ	23jumps 76.67% jumps 26.44% months	1 jumps 3.33% jumps 1.15% months	16 jumps 53.33% jumps $<$ 4.6% weekends

The number of jumps occure after no break and neither on the first nor the last day of a month:

- **SVLEPDEJ**: **3** (2013-03-26, 2014-06-20, 2015-02-10 with probabilities: 0.77927, 0.69244, 0.75563)
- **SVLEPNJ**: **4** (2013-03-26, 2014-06-19, 2014-06-20, 2015-02-10 with probabilities: 0.54477, 0.752265, 0.78491, 0.68966)

Forecasts - Contract price - SVLEPNJ model

Predictive distributions:



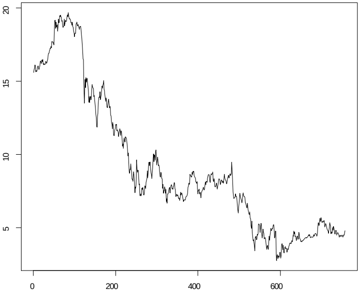
Forecast horizons:

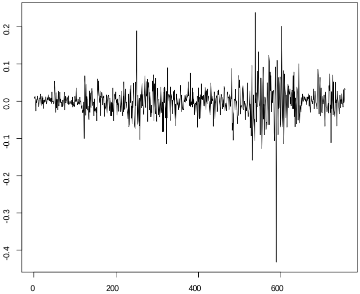
- 23.04.2014 (h=1, Thursday, the middle of the week)
- 27.04.2015 (h=3, Monday, the day after a break),
- 29.04.2015 (h=5, Wednesday the middle of the week),
- 01.05.2015 (h=7, Friday, the first day of the month),
- 05.05.2015 (h=8, Tuesday, the day after a break),
- 01.06.2015 (h=26, Monday, the first day of the month).

• observed prices, ▲ medians of predictive distributions

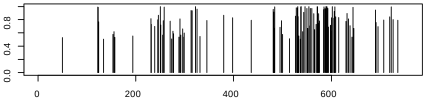
CO2 ALLOWANCES FUTURES CONTRACTS PRICES

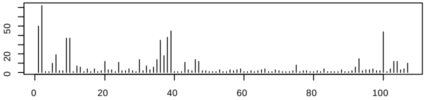
The ICE ECX futures. Daily logarithmic returns on CO_2 allowances futures expiring on December 16, 2013.
Data time range: January 3, 2011 - December 10, 2013 ($n = 756$)

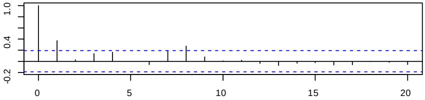




The ICE ECX futures contracts -EUAs- Analysis of jumps







The DEJD model

1. The posterior probabilities of jumps.
2. The values of the time elapsed between consecutive jumps with the number of a jump represented by the horizontal axis.
3. The autocorrelation function calculated for the series of times between consecutive jumps.

- The results may indicate the jump clustering phenomenon.

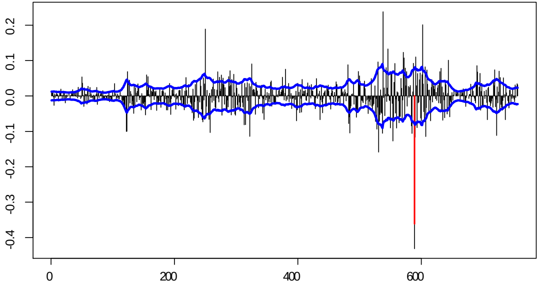
Analysis of jumps -EUAs

	$\Sigma_{D,U}$		$\Sigma_{D,U}$
$DEJD_{down}$	61	$VPT1_{down}$	7
$DEJD_{up}$	42	$VPT1_{up}$	7
$DEJD_{\Sigma=}$	**103**	$VPT1_{\Sigma=}$	**14**
$SVLEPDEJ_{down}$	1	$VPT2_{down}$	37
$SVLEPDEJ_{up}$	0	$VPT2_{down}$	36
$SVLEPDEJ_{\Sigma=}$	**1**	$VPT2_{\Sigma=}$	**73**
$SVLEPNJ_{down}$	1	RFR_{down}	7
$SVLEPNJ_{up}$	0	RFR_{up}	6
$SVLEPNJ_{\Sigma=}$	**1**	$RFR_{\Sigma=}$	**13**

Janczura et al. (2013):

- VPT1 (2.5% highest and 2.5% lowest returns are treated as jumps),
- VPT2 (10% highest and 10% lowest returns are treated as jumps),
- RFR (returns exceeding the mean return by three standard deviations are treated as jumps, with the jump values removed one by one ('recursive filter')).

Empirical Results - SVLEPDEJ model - EUAs



Logarithmic returns (black), posterior means of jump values (red) and SV (blue).

- **The jump occurs on 16.04.2015**: The European Parliament voted against the European Commission's proposal of delaying the auction of 900 million allowances from the first three years (2013-2015) of the 3rd ETS trading period (2013-2020) **(BACKLOADING)**.

Conclusions and Remarks

1. **Jump activity may vary over time so the assumption of a constant jump intensity under the jump-diffusion model appears no longer valid.**
2. **The jumps found under models with the SV component don't correspond to values below or above some fixed thresholds**.
3. **The stochastic volatility component strongly reduces the number of jumps and is a reason why the jump clustering phenomeon is rejected**.
4. **The frequency of positive jumps is higher than negative jumps (for the gas forwards series) and the jump value distribution is nonsymmetric around zero and separated from zero.**
5. **Most jumps in the gas contract prices are explained by the calendar effects**.

Thank you very much for your attention!

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