Pricing and hedging in regime switching stochastic volatility model: Application to electricity markets.

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Outline

Benth model: For all $t \in [0, T]$,

$$F_t(T) = F_0(T) \exp\left(m_t^T + \int_0^t \sigma_L(s) dW_s + \int_0^t \sigma_S(s) e^{-\lambda(T-s) dL_s} \right)$$

Studied in Goutte and al. [2], [3] using a quadratic approach.

Stochastic volatility:

- Complex volatility structure.
- Volatility varies over time: volatility increases when the time to maturity decreases (Samuelson hypothesis).

Regime switching: catches states of the world as

- "good" or "bad" economic.
- "on-peak" or "off-peak" time for electricity.

Let $(\omega, \mathcal{F}, P)$ be a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions for some fixed but arbitrary time horizon $T \in (0, \infty)$. The Regime Switching Stochastic Volatility model (RS-SV) is defined by

$$\begin{aligned} dS_t &= \mu(t, Y_t, X_t)S_t dt + Y_t S_t dW_t^1 \\ dY_t &= a(t, Y_t, X_t)dt + b(t, Y_t, X_t)dW_t^2 \end{aligned} \tag{1}$$

where

- W^1 and W^2 are two Brownian motion which are correlated as $d\langle W^1, W^2\rangle_t = \rho dt$.
- Y_t is a real stochastic process which is $\mathcal{F}_t$-adapted.
- X_t a continuous time homogeneous Markov chain on finite space $\mathcal{S} = \{1, 2, ..., N\}$.

We assume that the time invariant matrix Q denotes the generator $(q_{ij})_{i,j=1,\dots,m}$ of X, where q_{ij} is an infinitesimal intensity of X. Then, the semi-martingale decomposition for X is given by

$$X_t = X_0 + \int_0^t QX_s ds + M_t^X$$

where (M_t^X) is an $\mathbb{R}^N$-valued martingale with respect to the natural filtration generated by X under P.

Assumption 1

We will assume that we know all the trajectory of X, that is, $\mathcal{F}_T^X$ and that the Markov chain X is independent of S and Y.

In our model, there are three source of randomness: W^1, W^2 and X. Hence we will denote by $\mathbb{G}$, the filtration generated by W^1, W^2 and X. So the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]} := \sigma(W_t^1, W_t^2, X_t, 0 \leq t \leq T)$.

- **Hull-White**:

$$\begin{aligned}
\mu(t, Y_t, X_t) &= \mu Y_t, a(t, Y_t, X_t) = \frac{Y_t}{2}\left(\alpha - \frac{\beta^2}{2}\right), b(t, Y_t, X_t) = \frac{\beta}{2} Y_t \\
dS_t &= \mu Y_t S_t dt + Y_t S_t dW_t^1 \\
dY_t^2 &= \alpha Y_t^2 dt + \beta Y_t^2 dW_t^2
\end{aligned}$$

- **Stein-Stein**:

$$\begin{aligned}
\mu(t, Y_t, X_t) &= \mu Y_t, a(t, Y_t, X_t) = \alpha(\omega - Y_t), b(t, Y_t, X_t) = \beta \\
dS_t &= \mu Y_t S_t dt + Y_t S_t dW_t^1 \\
dY_t &= \alpha(\omega - Y_t)dt + \beta dW_t^2
\end{aligned}$$

- **Heston**:

$$\begin{aligned}
\mu(t, Y_t, X_t) &= \mu Y_t, a(t, Y_t, X_t) = \frac{4\kappa\theta - \sigma^2}{8Y_t} - \frac{\kappa}{2} Y_t, b(t, Y_t, X_t) = \frac{\sigma}{2} \\
dS_t &= \mu Y_t S_t dt + Y_t S_t dW_t^1 \\
dY_t^2 &= \kappa(\theta - Y_t^2)dt + \sigma Y_t dW_t^2 \\
2\kappa\theta &\geq \sigma^2
\end{aligned}$$

Outline

We are interested in the hedging of an European style contingent claims with an $\mathcal{F}_T$-measurable square integrable random payoff H based on the dynamics given by

$$\begin{aligned} dS_t &= \mu(t, Y_t, X_t) S_t dt + Y_t S_t dW_t^1 \\ dY_t &= a(t, Y_t, X_t) dt + b(t, Y_t, X_t) dW_t^2 \end{aligned}$$

Hence H is a function of time, S, Y and X:

$$\begin{aligned} H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{S} &\rightarrow \mathbb{R}^+ \\ (t, S_t, Y_t, X_t) &\mapsto h(t, S_t, Y_t, X_t) \end{aligned}$$

As example of payoff, we could take classical european Call option

$$H = h(S_T) := (S_T - K)^+$$

Definition 1

An **Hedging strategy** is a pair $\varphi = (v, \eta)$ such that $v = (v_t)_{t\in[0,T]}$ is a predictable process such that

$$\mathbb{E}\left[\int_0^T v_t^2 Y_t^2 S_t^2 dt\right] + \mathbb{E}\left[\left(\int_0^T |v_t||\mu(t, Y_t, X_t)|\right)^2\right] < \infty \qquad (2)$$

and $\eta = (\eta_t)_{t\in[0,T]}$ is an adapted process such that for all $t \in [0, T]$, $\mathbb{E}\left[\eta_t^2\right] < \infty$.

The hedging strategy φ defines a portfolio where v_t denotes the number of shares of the risky asset S held by the investor at time $t \in [0, T]$ and η_t denotes the amount invested at time t.

Definition 2

Given a hedging strategy φ, we call for all $t \in [0, T]$

- the **Value process** $V(\varphi)$, the right continuous process given by

$$V_t(\varphi) = v_t S_t + \eta_t \tag{3}$$

- the **Cost process** $C(\varphi)$, the process given by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t v_s dS_s \tag{4}$$

The quantity $\int_0^t v_s dS_s$ represents the hedging gains or losses up to time t following the hedging strategy φ. If $C(\varphi)$ is square integrable, then the **risk process** of φ is defined by

$$R_t(\varphi) := \mathbb{E}\left[(C_T(\varphi) - C_t(\varphi))^2 \,|\mathcal{G}_t \right] \tag{5}$$

The study of this minimization in a general semimartingale case is due to Schweizer [4]. Assume that $V_T(\varphi) = H$, the local risk minimization problem is to minimize the Risk process $R(\varphi)$. This require more specific assumptions on S. We assume that S can be decomposed as

$$S_t = S_0 + M_t + A_t$$

where M is a real valued locally square integrable local P-martingale null at 0 and A is a real valued adapted continuous process of finite variation also null at 0. We recall now the Definition of **Structure Condition** (SC). We say that S satisfies the (SC) if there exists a predictable process λ such that A is absolutely continuous with respect to $\langle M \rangle$ in the sense that

$$A_t = \int_0^t \lambda_s d\langle M \rangle_s$$

and such that the so called **mean variance tradeoff process** (MVT) K satisfied

$$K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s < \infty, \quad P - a.s.$$

Moreover Proposition 2.24 of Follmer and Schweizer in [1] shows that finding a locally risk minimizing strategy for a given contingent claim $H \in L^2(P)$ is equivalent to finding a decomposition of H of the form

$$H = H_0^{lr} + \int_0^T \xi_t^{lr} dS_t + L_T^{lr} \tag{6}$$

where H_0^{lr} is a constant, ξ^{lr} is a predictable process satisfying condition (2) and L^{lr} is a square integrable P-martingale null at 0 and strongly orthogonal to M (i.e. $L^{lr}M$ is a P-martingale). The representation (6) is usually referred to as the **Follmer-Schweizer (FS) decomposition** of H.

Once we have (6), then the desired hedging strategy φ^{lr} which is locally risk minimizing is then given for all $t \in [0, T]$ by

$$v^{lr_t} = \xi_t^{lr} \tag{7}$$

and

$$\eta_t^{lr} = V_t(\varphi^{lr}) - v_t^{lr} S_t \tag{8}$$

where

$$V_t(\varphi^{lr}) = C_t(\varphi^{lr}) + \int_0^t v_s^{lr} dS_s \tag{9}$$

with

$$C_t(\varphi^{lr}) = H_0^{lr} + L_t^{lr} \tag{10}$$

As is shown in [1] and [6] there exists a measure $\hat{P}$, the so called **minimal equivalent local martingale measure** (minimal ELMM), such that

$$V_t(\varphi^{lr}) = \hat{\mathbb{E}}\left[H|\mathcal{G}_t\right] \tag{11}$$

where $\hat{\mathbb{E}}$ denotes the conditional expectation under $\hat{P}$.
Theorem 1 of [1] allows us to construct uniquely $\hat{P}$ such that $\hat{P}$ exists if and only if for all $t \in [0, T]$

$$\hat{Z}_t = \exp\left(-\int_0^t \lambda_s dM_s - \frac{1}{2}\int_0^t \lambda_s^2 d\langle S\rangle_s\right) \tag{12}$$

is a square integrable martingale under P. Then

$$\frac{\hat{P}}{P} := \hat{Z}_T \in L^2(P)$$

defines a probability measure $\hat{P}$ equivalent to P which is in $\mathbb{P}$ since one easily verifies that $\hat{Z}S$ is a local P-martingale.

Let S, Y and X given as in model (1). The local risk minimizing hedging strategy can be obtained in two step.

(I) Determine $\hat{P}$ and the dynamic of (S, Y) under $\hat{P}$.

(II) Find the decomposition of H with respect to S under $\hat{P}$.

Then the optimal hedging strategy will defined by (7) and (8).

Finding $\hat{P}$:

According to previous subsection, the density process of the minimal ELMM $\hat{P}$ with respect to P is given by

$$\hat{Z}_t = \exp\left(-\int_0^t \lambda_s dM_s - \frac{1}{2}K_t\right)$$

Hence S is continuous, we need to determine the canonical decomposition $S_t = S_0 + M_t + \int_0^t \lambda_s d\langle M\rangle_s$ of S under P.

Proposition 1

Assume the regime stochastic volatility model (1) then we have that

$$M_t = \int_0^t S_s Y_s dW_s^1, A_t = \int_0^t \mu(s, Y_s, X_s) S_s ds \quad , \quad \langle M\rangle_t = \int_0^t S_s^2 Y_s^2 ds$$

$$\lambda_t = \frac{dA_t}{d\langle M\rangle_t} = \frac{\mu(t, Y_t, X_t)}{S_t Y_t^2} \quad \text{and} \quad K_t = \int_0^t \left(\frac{\mu(s, Y_s, X_s)}{Y_s}\right)^2 ds$$

$$\hat{Z} = \exp\left(-\int_0^t \frac{\mu(s, Y_s, X_s)}{Y_s} dW_s^1 - \frac{1}{2}\int_0^t \left(\frac{\mu(s, Y_s, X_s)}{Y_s}\right)^2 ds\right)$$

We are now able to determine the dynamic of the model under $\hat{P}$.

Proposition 2

Assume that $\hat{Z}$ is a true P-martingale, then the dynamic of the model (1) under $\hat{P}$ is given for all $t \in [0, T]$ by

$$\begin{aligned} dS_t &= Y_t S_t d\hat{W}_t^1 \\ dY_t &= \hat{a}(t, Y_t, X_t)dt + b(t, Y_t, X_t)d\hat{W}_t^2 \\ dY_t &= \hat{a}(t, Y_t, X_t)dt + b(t, Y_t, X_t)(\rho d\hat{W}_t^1 + \sqrt{1-\rho^2}d\hat{W}_t^3) \end{aligned} \quad (13)$$

with

$$\hat{a}(t, Y_t, X_t) = a(t, Y_t, X_t) - \frac{\rho}{Y_t}\mu(t, Y_t, X_t)b(t, Y_t, X_t) \quad (14)$$

and $\hat{W}^3$ is another Brownian motion independent of $\hat{W}^1$.

Decomposition of H with respect to S under $\hat{P}$:

Taking a contingent claim of the form $H = h(S_T)$ for some given function $h \to [0, \infty) \times \mathbb{R} \times \mathcal{S}$. Then finding the Galtchouk-Kunita-Watanabe decomposition of H under $\hat{P}$ reduces to solve a system of partial differential equation PDE if one exploits the Markovian structure.

Using the Markov property we can rewrite (11) in the form

$$V_t(\varphi^{lr}) = \hat{\mathbb{E}}\left[h(S_T)|\mathcal{F}_t\right] = \hat{v}(t, S_t, Y_t, X_t) \tag{15}$$

for some function $\hat{v}(t, s, y, x)$ defined on $[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{S}$.

Proposition 3:

We have that the conditional price of the contingent claim H is given by $\hat{v}$ as the solution to the system of partial differential equation given for all $i \in \mathcal{S}$ by

$$\begin{aligned} 0 &= \hat{v}_t(t,s,y,i) + \hat{a}(t,y,i)\hat{v}_y(t,s,y,i) \\ &+ \frac{1}{2}\left[s^2y^2\hat{v}_{ss}(t,s,y,i) + b^2(t,y,i)\hat{v}_{yy} + 2syb\rho\hat{v}_{sy}(t,s,y,i)\right] \\ &+ \sum_{j\neq i, j\in\mathcal{S}} q_{ij}\left(\hat{v}(t,s,y,j) - \hat{v}(t,s,y,i)\right) \end{aligned}$$

with terminal condition for all $i \in \mathcal{S}$,

$$\hat{v}(T,s,y,i) = h(s,y,i)$$

where $\hat{v}_t := \frac{\partial \hat{v}}{\partial t}$, $\hat{v}_y := \frac{\partial \hat{v}}{\partial Y}$, $\hat{v}_{ss} := \frac{\partial^2 \hat{v}}{\partial S^2}$, $\hat{v}_{yy} := \frac{\partial^2 \hat{v}}{\partial Y^2}$ and $\hat{v}_{sy} := \frac{\partial^2 \hat{v}}{\partial S \partial Y}$.

Hence for the particular case of european call option:
$\hat{v}(T,s,y,i) = (s_T(i) - K)^+$

We are now able to find the decomposition of H with respect to S under $\hat{P}$ and so the locally risk minimizing H-admissible strategy φ^{lr}.

Theorem 1:

For all $t \in [0, T]$ we have that the locally risk-minimizing hedging strategy of H, $\varphi^{lr} := (v^{lr}, \eta^{lr})$ is given by

$$v_t^{lr} = \hat{v}_s(u, S_u, Y_u, X_u) + \hat{v}_y(u, S_u, Y_u, X_u)\frac{\rho}{S_u Y_u} b(u, Y_u, X_u) \quad (16)$$

$$\eta_t^{lr} = V_t(\varphi^{lr}) - v_t^{lr} S_t \quad (17)$$

where $V_t(\varphi^{lr}) = V_0(\varphi^{lr}) + \int_0^t v_s^{lr} dS_s + L_t^{lr}$ and

$$L_t^{lr} = \int_0^t \sqrt{1-\rho^2}\hat{v}_y(u, S_u, Y_u, X_u) b(u, Y_u, X_u) d\hat{W}_u^3 + \int_0^t \int_{\mathcal{S}} \left[\hat{v}(u, S_u, Y_u, j) - \hat{v}(u, S_{u-}, Y_{u-}, X_{u-})\right] (\nu - \overline{\nu})(du, dj)$$

We can obtain a formulation of the conditional expected squared cost on the interval $[t, T]$ for the locally risk-minimizing strategy φ^{lr} which we recall that it is denoted by R_t^{lr}.

Proposition 4:

We have that for all $t \in [0, T]$ that the conditional expected squared cost on the interval $[t, T]$ for the locally risk-minimizing strategy φ^{lr} is given by

$$\begin{aligned} R_t^{lr} &= \mathbb{E}\left[\left(\int_t^T \sqrt{1-\rho^2}\hat{v}_y(u)b(u)d\hat{W}_u^3\right)^2 |\mathcal{F}_t\right] \\ &+2\mathbb{E}\left[\left(\int_t^T \sqrt{1-\rho^2}\hat{v}_y(u)b(u)d\hat{W}_u^3\right)\left(\int_t^T \int_{\mathcal{S}} \left[\hat{v}(u,j)-\hat{v}(u,X_{u-})\right](\nu-\overline{\nu})(du,dj)\right)|\mathcal{F}_t\right] \\ &+\mathbb{E}\left[\int_t^T \left[Q\hat{v}^2(u,X_{u-})-2\hat{v}(u,X_{u-})Q\hat{v}(u,X_{u-})\right]du|\mathcal{F}_t\right] \end{aligned}$$

Remark

To apply all the results about local risk minimizing hedging strategy , it remains to prove that $\hat{Z}$ is a true P-martingale and square integrable under P. A well-known sufficient condition for both is boundedness of the mean variance tradeoff process K uniformly in t and ω. This condition will be checked in our examples.

Heston model:

$$\begin{aligned}
\mu(t, Y_t, X_t) &= \mu(X_t) Y_t \quad with \quad \mu = (\mu_1, \ldots, \mu_N) \in \mathbb{R}^{card(\mathcal{S})} \\
a(t, Y_t, X_t) &= \frac{4\kappa(X_t)\theta(X_t) - \sigma(X_t)^2}{8 Y_t} - \frac{\kappa(X_t)}{2} Y_t, \quad with \quad \kappa = (\kappa_1, \ldots, \kappa
\\
b(t, Y_t, X_t) &= \frac{\sigma(X_t)}{2} \quad \text{and} \quad \rho = \rho_0 \in]-1, 1[
\end{aligned}$$

The constants κ_i, θ_i, σ_i are all nonnegative for all $i \in \mathcal{S}$ and for all $i \in \mathcal{S}$ that $\kappa_i \theta_i \geq \frac{1}{2}\sigma_i$. The model is then given by

$$\begin{aligned}
dS_t &= \mu(X_t) S_t dt + Y_t S_t dW_t^1 \\
dY_t^2 &= \kappa(X_t)(\theta(X_t) - Y_t^2) dt + \sigma(X_t) Y_t dW_t^2
\end{aligned}$$

The mean variance tradeoff process is then given by

$$K_t = \int_0^t \frac{\mu(t, Y_s, X_s)^2}{Y_s^2} ds = \int_0^t \frac{\mu^2(X_s) Y_s^2}{Y_s^2} ds = \int_0^t \mu^2(X_s) ds < \infty$$

Hence the MVT process K is deterministic so bounded uniformly in $t \in [0, T]$ and ω. This imply that $\hat{Z}$ is a P-martingale and so that we can apply all the result mentioned before.

Outline

Variance swap

A variance swap is a forward contract on annualized variance, which is the square of the realized annual volatility. Let Y_R^2 denote the realized annual stock variance over the life of the contract.

$$Y_R^2 = \frac{1}{T}\int_0^T Y_t^2 dt$$

Let K_v and N denote the delivery price for variance and the notional amount of the swap in dollars per annualized volatility point squared. Then, the payoff of the variance swap at expiration time T is given by $N(Y_R^2 - K_v)$. Intuitively, the buyer of the variance swap will receive N dollars for each point by which the realized annual variance Y_R^2 has exceeded the variance delivery price K_v.

Assume that we are under the ELMM $\hat{P}$.

$$\begin{aligned} dS_t &= Y_t S_t d\hat{W}_t^1 \\ dY_t &= \hat{a}(t, Y_t, X_t)dt + b(t, Y_t, X_t)d\hat{W}^2{}_t \end{aligned}$$

with $\hat{a}(t, Y_t, X_t) = a(t, Y_t, X_t) - \frac{\rho}{Y_t}\mu(t, Y_t, X_t)b(t, Y_t, X_t)$. In particular, given $\mathcal{F}_T^X$, **the conditional price of the variance swap** $P(X)$ is given by

$$\begin{aligned} P(X) &= \hat{\mathbb{E}}\left[\exp\left(-\int_0^T r_t dt\right) N(Y_R^2 - K_v)|\mathcal{F}_T^X\right] \\ &= \exp\left(-\int_0^T r_t dt\right) N\hat{\mathbb{E}}\left[Y_R^2|\mathcal{F}_T^X\right] - \exp\left(-\int_0^T r_t dt\right) NK_v \end{aligned}$$

where N is the notional amount in dollars.

Itô formula gives that for all $t \in [0, T]$

$$\frac{d\hat{\mathbb{E}}^X\left[Y_t^2\right]}{dt} = \hat{\mathbb{E}}^X\left[2Y_t\hat{a}(t, Y_t, X_t) + b^2(t, Y_t, X_t)\right] \tag{18}$$

Assumption 2:

Assume that we know the solution of equation (18) which we denote for all $t \in [0, T]$ by: $y(t, Y_t, X_t)$.

Proposition 5:

Under Assumption 2, we have for all $t \in [0, T]$ that the variance swap price P(X) is given by

$$P(X) = \exp\left(-\int_0^T r_t dt\right) N\left(\frac{1}{T}\int_0^T y(t, Y_t, X_t)dt - K_v\right) \tag{19}$$

Heston Model

$\hat{a}(t, Y_t, X_t) = \frac{4\kappa(X_t)\theta(X_t) - \sigma(X_t)^2}{8Y_t} - \frac{\kappa(X_t)}{2} Y_t$ and $b(t, Y_t, X_t) = \frac{\sigma(X_t)}{2}$. Then (18) becomes

$$\frac{d\hat{\mathbb{E}}^X\left[Y_t^2\right]}{dt} = \kappa(X_t)\left(\theta(X_t) - \hat{\mathbb{E}}^X\left[Y_t^2\right]\right)$$

Let $y_t := \hat{\mathbb{E}}^X\left[Y_t^2\right]$, then we need to solve the differential equation

$$\frac{dy_t}{d_t} = \kappa(X_t)\left(\theta(X_t) - y_t\right)$$

The solution of this differential equation is given for all $t \in [0, T]$ by

$$y_t = y_0 \exp\left(-\int_0^t \kappa_s ds\right) + \frac{\int_0^t \left(\exp\left(\int_0^s \kappa_u du\right)\kappa_s\theta_s\right) ds}{\exp\left(\int_0^t \kappa_s ds\right)}$$

Hence we find

$$\hat{\mathbb{E}}^X\left[Y_t^2\right] = Y_0^2 \exp\left(-\int_0^t \kappa_s ds\right) + \frac{\int_0^t \left(\exp\left(\int_0^s \kappa_u du\right)\kappa_s\theta_s\right) ds}{\exp\left(\int_0^t \kappa_s ds\right)} \qquad (20)$$

We can now obtain the variance swap price applying Proposition 5:

$$P(X) \quad = \quad \exp\left(-\int_0^T r_t dt\right) N\left[\frac{1}{T}\int_0^T \left(Y_0^2 \exp\left(-\int_0^t \kappa_s ds\right) + \frac{\int_0^t \left(\exp\left(\int_0^s \kappa_u du\right)\kappa_s\theta_s\right) ds}{\exp\left(\int_0^t \kappa_s ds\right)}\right) dt - K_v\right]$$

Volatility swap

We recall that the realized annual stock variance over the life of the contract is given by

$$Y_R^2 := \frac{1}{T}\int_0^T Y_t^2 dt$$

and depend on the Markov chain X. Denote by $I_t = \int_0^t Y_s^2 ds$ the accumulated variance where the process Y^2 is solution of the stochastic differential equation (SDE) given by

$$dY_t^2 = \left(2Y_t\hat{a}(t, Y_t, X_t) + b^2(t, Y_t, X_t)\right) dt + 2Y_t b(t, Y_t, X_t) d\hat{W}_t^2 \quad (21)$$

Hence I_t is the solution of the SDE given by $dI_t = Y_t^2 dt$. Let define by E_t^T the expectation at time $t \in [0, T]$ given X by

$$E_t^T = \hat{\mathbb{E}}_t\left[\frac{1}{T}\int_0^T Y_s^2 ds | \mathcal{F}_T^X\right] = \hat{\mathbb{E}}_t^X\left[\frac{1}{T}\int_0^T Y_s^2 ds\right] \quad (22)$$

Hence $(E_t^T)_{t\in[0,T]}$ depends on the variance process Y^2 of the underlying asset and on the Markov chain X. We call by fair variance strike price the quantity K_v^* which is such that the variance swap price $P(X)$ vanishes:

$$\begin{aligned} P(X) &= \hat{\mathbb{E}}\left[\exp\left(-\int_0^T r_t dt\right) N(Y_R^2 - K_v^*)|\mathcal{F}_T^X\right] \\ &= \hat{\mathbb{E}}^X\left[\exp\left(-\int_0^T r_t dt\right) N(Y_R^2 - K_v^*)\right] = 0 \end{aligned}$$

Then we have that $K_v^* = \hat{\mathbb{E}}\left[Y_R^2|\mathcal{F}_T^X\right]$ and for time $t = 0$, we obtain that $E_0^T = K_v^*$. We define now the forward price process Z_t^T as

$$Z_t^T = \hat{\mathbb{E}}_t\left[\sqrt{\frac{1}{T}\int_0^T Y_s^2 ds}|\mathcal{F}_T^X\right] = \hat{\mathbb{E}}_t^X\left[\sqrt{\frac{1}{T}\int_0^T Y_s^2 ds}\right] \qquad (23)$$

Proposition 6:

The forward price process Z_t^T can be expressed as a function $F(t, Y_t^2, X_t, I_t)$ and is the solution of the system of stochastic differential equation given by

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial I} Y_t^2 + \frac{\partial F}{\partial Y^2} \left(2 Y_t \hat{a}_t + b_t^2\right) dt + \frac{1}{2} \frac{\partial^2 F}{\partial (Y^2)^2} 4 Y_t^2 b_t^2 dt + \langle F, QX_t \rangle = 0 (2$$

with boundary condition given by $F(T, Y_T^2, X_T, I_T) = \sqrt{\frac{I_T}{T}}$.

Outline

To solve system of SDE of Proposition 3, we can use numerical grid using explicit scheme. For all $k \in \mathcal{S}$ we construct a grid of size $[0, \overline{S}] \times [0, \overline{Y}]$. The corresponding discretization will contains $I+1$ nodes in S direction and $J+1$ nodes in Y direction. Then all partial differentiations could be stated as following:

$$
\begin{aligned}
\hat{v}_y(k) &= \frac{\hat{v}^n_{i,j+1}(k) - \hat{v}^n_{i,j-1}}{2\Delta Y} \\
\hat{v}_{ss}(k) &= \frac{\hat{v}^n_{i+1,j}(k) - 2\hat{v}^n_{i,j}(k) + \hat{v}^n_{i-1,j}(k)}{(\Delta S)^2} \\
\hat{v}_{yy}(k) &= \frac{\hat{v}^n_{i,j+1}(k) - 2\hat{v}^n_{i,j}(k) + \hat{v}^n_{i,j-1}(k)}{(\Delta Y)^2} \\
\hat{v}_{sy}(k) &= \frac{\hat{v}^n_{i+1,j+1}(k) + \hat{v}^n_{i-1,j-1}(k) - \hat{v}^n_{i-1,j+1}(k) - \hat{v}^n_{i+1,j-1}(k)}{4\Delta S \Delta Y}
\end{aligned}
$$

and

$$
\hat{v}_t(k) = \frac{\hat{v}^{n+1}_{i,j} - \hat{v}^n_{i,j}}{\Delta t}
$$

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